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Nonsmooth multiobjective continuous-time problems with generalized invexity

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Abstract A nonsmooth multiobjective continuous-time problem is considered. The definition of invexity and its generalizations for continuous-time functions are extended. Then, optimality conditions under generalized invexity assumptions are established. Subsequently, these optimality conditions are utilized as a basis for formulating dual problems. Duality results are also obtained for Wolfe as well as Mond-Weir type dual, using generalized invexity on the functions involved.

Keywords Continuous-time problems · Multiobjective programming · Efficient solution · Duality · Nonsmooth analysis

Mathematics Subject Classification (2000) 90C46 · 90C29 · 49J52

1 Introduction

The relationship between mathematical programming and classical calculous of variation was explored and extended by Hanson [6]. Since then, variational programming problems have attracted some attention in the literature (see, e.g. [1,8,12]). Optimality conditions and duality results were obtained for a variational problem by Mond and Hanson [9] under convexity assumptions. In mathematical programming, the Kuhn-Tucker conditions are sufficient for optimality if the functions involved are convex. In the last few years, attempts have been made to weaker the convexity hypotheses. As it is known, invexity has been introduced in optimization theory by Hanson, see [7], as a substitute for convexity in constraint optimization. Subsequently, Vial [13] has introduced ρ-convex functions and he used this concept to

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obtain some duality results. Invexity was extended to varaitional problems by Mond et al. [10].

In this paper, we introduce the definition of ρ-invexity in the continuous time context. Then, optimality conditions for nonsmooth multiobjective continuous-time problems are established under various ρ-invexity restrictions on the component of the functions describing constraints and the objectives functions.

This work is organized as follows. In Sect. 2, we recall some basic definitions of nonsmooth analysis and we present a numbers of definitions which will be needed in the sequel. In Sect. 3, we establish the nonsmooth optimality conditions for a class of nonsmooth multiobjective continuous-time problems. In Sect. 4, we formulate and discuss two duality type problems and prove appropriate duality theorems.

2 Preliminary

In this work we introduce the following multiobjective continuous-time problem:

$$(MCT) \quad \min\Big[\int_0^T f_1(t, x(t))dt, \cdots, \int_0^T f_r(t, x(t))dt\Big]$$
$$\text{subject to}$$
$$g_i(t, x(t)) \leq 0, \qquad a.e. \quad t \in [0, T],$$
$$i \in M = \{1, 2, ..., m\} \quad x \in X.$$

Here X is a nonempty open convex subset of the Banach space $L^n_\infty[0, T]$ of all n-dimensional vector-valued Lebesgue measurable essentially bounded functions defined on the compact interval $[0, T] \subset \mathbb{R}$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1\leq j\leq n} \operatorname{ess\,sup}\{|x_j(t)|, 0 \leq t \leq T\},$$

where for each $t \in [0, T]$, $x_j(t)$ is the jth component of $x(t) \in \mathbb{R}^n$. We define $g(t, x(t)) = G(x)(t)$ and $f_i(t, x(t)) = \phi_i(x)(t)$, $i \in L = \{1, 2, ..., r\}$, where $G(\cdot)$ is a map from X into the normed space $\Lambda^m_1[0, T]$ of all Lebesgue measurable essentially bounded m-dimensional vector functions defined on $[0, T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1\leq i\leq m} \int_0^T |y_i(t)|dt,$$

and ϕ_i is a map from X into the normed space $\Lambda^1_1[0, T]$.
Let F_p be the set of feasible solutions to (MCT),

$$F_P = \{x \in X : g_i(t, x(t)) \leq 0, \quad a.e\ t \in [0, T], i \in M\}.$$

We assume that each functions $t \to f_i(t, x(t))$ and $t \to g_i(t, x(t))$ are Lebesgue measurable and integrable for all $x \in X$. We also assume that the functions $f_i(t, x(t))$ and $g_j(t, x(t))$ are locally Lipschitz on X throughout $[0, T]$.

We recall basic concepts and tools from nonsmooth analysis. Most of the material included here can be found in [4,5]. Let Y be a Banach space and $F : Y \to \mathbb{R}$ be a locally Lipshitz function; i.e., for each $y \in Y$, there exists $\varepsilon > 0$ and a constant $K > 0$, such that

$$|F(x_1) - F(x_2)| \leq K\|x_1 - x_2\| \quad \forall x_1, x_2 \in y + \epsilon B.$$

where B is the open unit ball of Y.

The Clarke generalized directional derivative of f at x in the direction of a given $d \in Y$, denote by $F^0(x; d)$, is defined by

$$F^0(x; d) := \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{F(y + \lambda d) - F(y)}{\lambda}.$$

The generalized gradient of F at x, denoted by $\partial_c F(x)$, is defined by

$$\partial_c F(x) := \{\xi \in Y^* : \langle \xi, d \rangle \leq F^0(x; d) \quad \forall d \in Y\}.$$

Here, Y^* denotes the dual space of continuous linear functionals on Y and $\langle \cdot, \cdot \rangle : Y^* \times Y \to \mathbb{R}$ is the duality pairing.

Now, assume $\bar{x} \in X$ and $h \in L^n_\infty[0, T]$. The continuous Clarke generalized directional derivative of f_i is defined by

$$f_i^0(t, \bar{x}(t); h(t)) := \phi_i^0(\bar{x}; h)(t) := \limsup_{\substack{y \to \bar{x} \\ \lambda \downarrow 0}} \frac{\phi_i(y + \lambda h)(t) - \phi_i(y)(t)}{\lambda},$$

a.e. $t \in [0, T]$.

We recall the integration of multifunctions.

Given a multifunction $H : [0, T] \to \mathbb{R}^n$, denote by F(H) the following set:

$$F(H) = \{f \in L_1^n[0, T], f(t) \in H(t) \quad a.e. \quad t \in [0, T]\}.$$

We define the integral of H, denoted by $\int_0^T H(t)dt$, as the following subset of $\mathbb{R}^n$:

$$\int_0^T H(t)dt := \left\{ \int_0^T f(t)dt : f \in F(H) \right\}.$$

We introduce invexity notion in the continuous-time context. Let f be a real function on $[0, T] \times X$ and suppose that $f(t, .)$ is locally Lipschitz on X throughout $[0, T]$. Assume that there exists a function $\eta : [0, T] \times X \times X \to \mathbb{R}$ such that the function $t \to \eta(t, x(t), \bar{x}(t))$ is in $L^n_\infty[0, T]$, $\eta(t, x(t), x(t)) = 0$.

Definition 2.1 The functional $F(x) = \int_0^T f(t, x(t))dt$ is said to be

(i) invex at $\bar{x}$, with respect to η, if for all $x \in X$,

$$\int_0^T f(t, x(t))dt - \int_0^T f(t, \bar{x}(t))dt \geq \int_0^T f^0(t, \bar{x}(t); \eta(t, x(t), \bar{x}(t))dt,$$

(ii) is strictly invex if the above inequality is strict for $x(t) \neq \bar{x}(t)$.

Assume that there exist function $\eta : I \times X \times X \to \mathbb{R}$ with $\eta(t, x, x) = 0$ and $d(., ., .)$ is a pseudometric and $\rho \in R$.

Definition 2.2 The functional $F(x)$ is said to be ρ-pseudoinvex at $\bar{x}$ with respect to function η, if for all $x \in X$,

$$\int_0^T f^0(t, \bar{x}(t); \eta(t, x(t), \bar{x}(t)) \geq -\rho \int_0^T d^2(t, x(t), \bar{x}(t))dt$$
$$\Rightarrow F(x) \geq F(\bar{x}).$$

Definition 2.3 The functional $F(x)$ is said to ρ-strictly pseudoinvex at $\bar{x}$ with respect to function η, if for all $x \in X$,

$$\int_0^T f^0(t, \bar{x}(t); \eta(t, x, \bar{x}) \geq -\rho \int_0^T d^2(t, x(t), \bar{x}(t))dt$$
$$\Rightarrow F(x) > F(\bar{x}).$$

Definition 2.4 The functional $F(x)$ is said to be ρ-quasiinvex at $\bar{x}$ with respect to function η, if for all $x \in X$,

$$F(x) \leq F(\bar{x}) \Rightarrow \int_a^b f^0(t, \bar{x}(t); \eta(t, x(t), x\bar{(t)})dt \leq -\rho \int_0^T d^2(t, x(t), \bar{x}(t))dt.$$

The functional $F(x)$ is said to be ρ-strictly quasiinvex at $\bar{x}$ with respect to function η, if for all $x \in X$,

$$F(x) \leq F(\bar{x}) \Rightarrow \int_a^b f^0(t, \bar{x}(t); \eta(t, x(t), x\bar{(t)})dt < -\rho \int_0^T d^2(t, x(t), \bar{x}(t))dt.$$

We use the acronyms $\rho - PIX$, $\rho - SPIX$, $\rho - QIX$, $\rho - SQIX$, for $F(.)$ when it is ρ-pseudoinvex, ρ- strictly pseudoinvex, ρ-quasiinvex and ρ-strictly quasiinvex, respectively, at each point of X

Definition 2.5 [3] A feasible solution x^* for MCT is said to be efficient for MCT if there is no other feasible x for MCT such that

$$\int_0^T f_i(t, x(t))dt < \int_0^T f_i(t, x^*(t))dt, \quad \text{for some} \quad i \in M,$$
$$\int_0^T f_j(t, x(t))dt \leq \int_0^T f_j(t, x^*(t))dt, \quad \text{for all} \quad j \in M.$$

Below we introduce two examples which show that in general, one can find a ρ-pseudoinvex or ρ-quasiinvex function which is not invex function with respect to the same η. This implies that our conditions are more general than invexity.

In the following example we consider a function which is ρ-pseudoinvex but it is not invex with respect to the function η defined by $\eta(x(t), y(t)) = x(t) - y(t)$.

Example 2.6 Define the function $f : [0, 1] \times [-1, 1] \to \mathbb{R}$ by

$$f(t, x(t)) = \begin{cases} \alpha x(t), & 0 \leq x \leq 1, \\ x(t), & -1 \leq x < 0, \end{cases}$$

where $x : [0, 1] \to [-1, 1]$ is defined by $x(t) = tx$, $x \in \mathbb{R}$ and $0 < \alpha < 1$.

It is easy to verify that $F(x) = \int_0^1 f(t, x(t))dt$ is ρ-pseudoinvex at $x^*(t) \equiv 0$ with $\rho < 0$ and $d(x, 0) = |x|^{\frac{1}{2}}$.

Now we present an example of a ρ-quasinvex function which is not invex with respect to the function η defined by $\eta(x(t), y(t)) = x(t) - y(t)$.

Example 2.7 Let the function $f(t, x(t)) : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t, x(t)) = \begin{cases} x^3(t), & x \geq 1, \\ x(t), & x < 1, \end{cases}$$

where $x(t)$ is defined by $x(t) = tx$, $x \in \mathbb{R}$.

At $x^*(t) = t$, the functional $F(x) = \int_0^1 f(t, x(t))dt$ is ρ-quasiinvex with respect to the function η at $x^*(t) = t$ with $\rho < 0$ and $d(x, 0) = |x|^{\frac{1}{2}}$.

3 Optimality conditions

In this section we discuss optimality conditions for (MCT) under various generalized ρ-invexity conditions. To deduce our main results, the following lemmas are necessary.

Lemma 3.1 [3] *A point $x^* \in F_P$ is an efficient solution for (MCT) if and only if x^* solves $P_k(x^*)$ for all $k = 1, 2, \ldots, r$, defined as*

$$\begin{array}{ll} & \min \int_0^T f_k(t, x(t))dt \\ P_k(x^*) & \textit{subject to} \\ & g_i(t, x(t)) \leq 0 \quad i \in M \quad a.e. \quad t \in [0, T] \\ & \int_0^T f_j(t, x(t))dt \leq \int_0^T f_j(t, x^*(t))dt, \end{array}$$

for all $j \in L = \{1, 2, \ldots, r\}$, $j \neq k$.

Consider the following constraint qualification(CQ) from [2],

$$(CQ) \quad \cap_{i \in M} K(g_i, \bar{x}) \neq \emptyset,$$

where

$$K(g_i, \bar{x}) = \{h \in L^n_\infty[0, T] : g_i^0(t, \bar{x}(t); h(t)) < 0, \quad \text{a.e. } t \in A_i(\bar{x}), i \in M\}$$

and $A_i(\bar{x}) = \{t \in [0, T] : g_i(t, \bar{x}(t)) = 0\}$.

Lemma 3.2 [11] *If $\bar{x}$ is an efficient solution of (MCT), and $P_k(\bar{x})$ satisfies the constraint qualification (CQ) at $\bar{x}$ for some k, then there exist $\tau^0 \in \mathbb{R}^r$ and piecewise smooth function $\lambda^0 : I \to \mathbb{R}^k$ satisfying the following:*

$$0 \in \int_0^T \left(\sum_{i=1}^r \tau_i^0 \partial_c f_i(t, \bar{x}(t)) + \sum_{j \in M} \lambda_j^0(t) \partial_c g_j(t, \bar{x}(t)) \right) dt \tag{1}$$

$$\sum_{j=1}^m \lambda_j^0 g_j(t, \bar{x}(t)) = 0, \lambda_j^0(t) \geq 0, \quad a.e. \quad t \in [0, T], \quad \sum_{i=1}^r \tau_i^0 = 1, \tau_i^0 \geq 0. \tag{2}$$

Theorem 3.3 *Suppose that there exist a feasible solution x^* for (MCT) and $\tau \in \mathbb{R}^r, \lambda \in L_\infty^m[0, T]$ such that:*

$$0 \in \int_0^T \left(\sum_{i=1}^r \tau_i \partial_c f_i(t, x^*(t)) + \sum_{j \in M} \lambda_j(t) \partial_c g_j(t, x^*(t)) \right) dt \tag{3}$$

$$\sum_{j=1}^m \lambda_j(t) g_j(t, x^*(t)) = 0, \quad \lambda_j(t) \geq 0, \quad \sum_{i=1}^r \tau_i = 1, \quad \tau_i \geq 0. \tag{4}$$

If $\int_0^T f_i(t, x(t))dt, i \in L$, are ρ_i-SQIX and $\int_0^T \lambda_i(t) g_i(t, x(t)), i \in M$ are σ_i-SQIX at x^* with the same $\eta(t, x(t), x^*(t))$ for all functions, with $\sum(\tau_i \rho_i + \sigma_i) \geq 0$, then x^* is an efficient solution for (MCT).

Proof Suppose that x^* is not an efficient solution for (MCT). Then there exists $x \in F_p$, such that

$$\int_0^T f_i(t, x(t))dt < \int_0^T f_i(t, x^*(t))dt, \quad \text{for some i } \in L,$$

$$\int_0^T f_i(t, x(t))dt \leq \int_0^T f_i(t, x^*(t))dt. \quad \text{for all i } \in L.$$

It follows by (4) that

$$\lambda_j(t) g_j(t, x(t)) \leq 0 = \lambda_j(t) g_j(t, x^*(t)) \quad \text{a.e. in } [0, T], \, j \in M.$$

By quasiinvexity assumptions, we have

$$\int_0^T \{\lambda_j(t) g_j^0(t, x^*(t); \eta(t, x(t), x^*(t))\} dt < -\sigma_j \int_0^T d^2(t, x(t), x^*(t)) dt. \tag{5}$$

$$\int_0^T f_i^0(t, x^*(t); \eta(t, x(t), x^*(t)) dt < -\rho_i \int_0^T d^2(t, x(t), x^*(t)) dt. \tag{6}$$

Now from (5 − 6) it follows that

$$\int_0^T \left\{ \sum_{i=1}^r \tau_i f_i^0(t, x^*(t); \eta(t, x(t), x^*(t)) + \sum_{j=1}^m \lambda_j(t) g_j^0(t, x^*(t); \eta(t, x(t), x^*(t)) \right\} dt$$

$$< -\sum (\tau_i \rho_i + \sigma_i) \int_0^T d^2(t, x(t), x^*(t))dt < 0,$$

which contradicts (3). Therefore, we conclude that x^* is an efficient solution of (MCT). □

Remark 3.4 Theorem 3.3 also holds under the following different types of assumptions;

(a) $\int_0^T f_i(t, x(t))dt$ is ρ_i-QIX at x^* with respect to η for all $i \in L$ and $\int_0^T w(t)^T g^0(t, x(t))dt$ is σ-SQIX at x^* with respect to the same η and $\sum \tau_i \rho_i + \sigma \geq 0$.

(b) $\int_0^T f_i(t, x(t))dt$ is ρ_i-QIX at x^* with respect to functions η for all $i \in L$ and $\int_0^T w(t)^T g^0(t, x(t)dt$ is σ-QIX at x^* with respect to the same η and $\sum \tau_i \rho_i + \sigma > 0$.

4 Duality theorems

In this section two duals for (MCT) are proposed and duality relationships are established under generalized ρ-invexity assumptions:

4.1 Wolf dual(WD)

.

$$\max \left[\int_0^T \{(f_1(t, u(t)) + w(t)^T g(t, u(t))\}dt, ..., \right.$$
$$\left. \int_0^T \{(f_r(t, u(t)) + w(t)^T g(t, u(t))\}dt \right]$$

subject to

$$0 \in \int_0^T \{\sum_{i=1}^r \tau_i \partial_c f_i(t, u(t)) + w(t)^T \partial_c g(t, u(t))\}dt \quad t \in [0, T], \tag{7}$$

$$\int_0^T w(t)^T g(t, u(t))dt \geq 0, \tag{8}$$

$$\sum_{i=1}^{r} \tau_i = 1; \tau_i \geq 0, \quad w_j(t) \geq 0, \quad j \in M \quad a.e. \quad t \in [0, T]. \tag{9}$$

4.2 Mond-Weir dual (MWD)

$$\max \left[\int_0^T f_1(t, u(t))dt, \ldots, \int_0^T f_2(t, u(t))dt, \ldots, \int_0^T f_r(t, u(t))dt \right]$$

subject to

$$0 \in \int_0^T \left\{ \sum_{i=1}^{r} \tau_i \partial_c f_i(t, u(t)) + w(t)^T \partial_c g(t, u(t)) \right\} dt, \quad t \in [0, T] \tag{10}$$

$$\int_0^T w(t)^T g(t, u(t))dt \geq 0, \tag{11}$$

$$\sum_{i=1}^{r} \tau_i = 1, \quad \tau_i \geq 0, i \in L \qquad w_j(t) \geq 0, \quad j \in M, \quad a.e. \quad t \in [0, T].$$

Theorem 4.1 (Weak Duality) *Assume that for all feasible solutions x for (MCT) and all feasible solutions* (u, τ, w) *for* (WD)*:*

(i) $\int_0^T \{f_i(t, x(t)) + w(t)^T g(t, x(t))\}dt$ *is* ρ_i*-SPIX with respect to* η *for all* $i \in L$ *and* $\sum_i^r \tau_i \rho_i \geq 0$. *Then, the following can not be hold:*

$$\int_0^T f_i(t, x(t))dt < \int_0^T \{f_i(t, u(t))dt + w(t)^T g(t, u(t))\}dt, \tag{12}$$

for some i $\in M$

$$\int_b^T f_j(t, x(t))dt \leq \int_0^T \{f_j(t, u(t)) + w(t)^T g(t, u(t))\}dt, \quad \forall j \in M \tag{13}$$

Proof Suppose contrary to the result of theorem that (12) and (13) hold. Since x is a feasible solution for MCT and (u, τ, w) is a feasible solution for WD, it follows that

$$\int_0^T \{f_i(t, x(t)) + w(t)^T g(t, x(t))\}dt < \int_0^T \{f_i(t, u(t)) + w(t)^T g(t, u(t))\}dt, \tag{14}$$

for some $i \in L$,

$$\int_0^T \{f_j(t, x(t)) + w(t)^T g(t, x(t))\}dt \leq \int_0^T \{f_j(t, u(t)) + w(t)^T g(t, u(t))\}dt, \tag{15}$$

for all $j \in L$.

Using (i) and (14-15) we get

$$\int_0^T \{f_i^0(t,x(t));\eta(t,x(t),u(t)) + w(t)^T g^0(t,u(t));\eta(t,x(t),u(t))\}dt \tag{16}$$

$$< -\rho_i \int_0^T d^2(t,x(t),u(t))dt \quad \forall i \in L.$$

Multiplying each inequality of (16) by $\tau_i \geq 0, i = 1, 2, \ldots, r$ and adding,

$$\int_0^T \left[\sum_{i=1}^r \tau_i f_i^0(t,x(t));\eta(t,x(t),u(t)) + w(t)^T g^0(t,u(t));\eta(t,x(t),u(t))\right] dt \tag{17}$$

$$< -(\sum_i^r \tau_i \rho_i) \int_0^T d^2(t,x(t),u(t))dt < 0.$$

This contradicts (7). □

Remark 4.2 The weak duality theorem also holds under the following different types of assumptions;

(ii) $\tau_i > 0, \int_0^T\{f_i(t,x(t)) + w(t)^T g(t,x(t))\}dt$, is ρ_i-PIX with respect to function η for all $i \in L$ and $\sum_{i=1}^r \tau_i \rho_i \geq 0$.

(iiii) $\int_0^T\{f_i(t,x(t)) + w(t)^T g(t,x(t))dt$, is ρ_i-QIX with respect to function η for all $i \in L$ and $\sum_{i=1}^r \tau_i \rho_i > 0$.

Corollary 4.3 *Assume that weak duality theorem 4.1 holds between (MCT) and (WD). If u^0 is feasible for (MCT), and (u^0, τ^0, ω^0) is feasible for (WD) with $w^0(t)^T g(t,u^0(t)) = 0$. Then, u^0 is an efficient for (MCT) and (u^0, τ^0, ω^0) is an efficient solution for (WD).*

Proof Suppose u^0 is not efficient for (MCT). Then there exists a feasible solution x for (MCT) such that

$$\int_0^T f_i(t,x(t))dt < \int_0^T f_i(t,u^0(t))dt, \quad \text{for some} \quad i \in L,$$

$$\int_0^T f_i(t,x(t))dt \leq \int_0^T f_i(t,u^0(t))dt, \quad \text{for all} \quad i \in L.$$

Since $w^0(t)^T g(t,u^0(t)) = 0$ we obtain

$$\int_0^T f_i(t,x(t))dt < \int_0^T f_i(t,u^0(t))dt + w(t)^T g(t,u^0(t))dt, \quad \text{for some} \quad i \in L,$$

$$\int_0^T f_i(t, x(t))dt < \int_0^T \{f_i(t, u^0(t))dt + w(t)^T g(t, u^0)\}dt, \quad \text{for all} \quad i \in L.$$

This contradicts weak duality. Hence u^0 is an efficient solution for (MCT). Now suppose that (u^0, τ^0, ω^0) is not an efficient solution for (WD). Then there exists (u, τ, ω) feasible for (WD) such that

$$\int_0^T \{f_i(t, u(t)) + w(t)^T g(t, u(t))\}dt > \int_0^T \{f_i(t, u^0(t)) + w^0(t)^T g(t, u^0)\}dt,$$

for some $i \in L$,

$$\int_0^T \{f_i(t, u(t)) + w(t)^T g(t, u(t))\}dt \geq \int_0^T \{f_i(t, u^0(t)) + w^0(t)^T g(t, u^0)\}dt,$$

for all $i \in L$. Since $w^0(t)^T g(t, u^0) = 0$, then

$$\int_0^T \{f_i(t, u(t))dt + w(t)^T g(t, u(t))dt > \int_0^T f_i(t, u^0(t))dt,$$

for some $i \in L$,

$$\int_0^T \{f_i(t, u(t))dt + w(t)^T g(t, u(t))dt \geq \int_0^T f_i(t, u^0(t))dt,$$

for all $i \in L$. This contradicts weak duality. Hence (u^0, τ^0, ω^0) is an efficient solution for (WD). □

Theorem 4.4 (Strong duality) *Let u^0 be an efficient solution for (MCT) and assume that u^0 satisfies the constraint qualification (CQ) for $P_k(u^0)$ for at least one $k \in L$. Then, there exist $\tau^0 \in \mathbb{R}^r$ and piecewise smooth function $w^0 : I \to \mathbb{R}^p$ such that (u^0, τ^0, w^0) is feasible for (WD) and $w^0(t)^T g(t, u^0(t)) = 0$. If weak duality also holds between (MCT) and (WD) then, (u^0, τ^0, w^0) is an efficient solution for (WD).*

Proof Since u^0 satisfies the constraint qualification for at least one k, it follows from Lemma 3.2 that there exist $\tau^0 \in \mathbb{R}^r$ and piecewise smooth function $w^0 : I \to \mathbb{R}^m$ such that

$$0 \in \int_0^T \left\{ \sum_{i=1}^r \tau_i^0 (\partial_c f_i(t, u^0(t)) + w^0(t)^T \partial_c g(t, u^0(t))) \right\} dt, \tag{18}$$

$$w^0(t)^T g(t, u^0(t)) = 0, \quad a.e. \quad t \in [0, T] \tag{19}$$

$$w^0(t) \geq 0, \tau_i^0 \geq 0, i = 1, \ldots, r \quad \sum_{i=1}^r \tau_i^0 = 1. \tag{20}$$

Now it follows from $(18-19)$, that (u^0, τ^0, w^0) is feasible for (WD). Also $w^0(t)^T g(t, u^0(t)) = 0$ and weak duality holds between (MCT) and (WD). Thus the result follows from Corollary 4.3. □

4.3 Duality between MCT and MWD

Theorem 4.5 (Weak Duality) *Assume that for all feasible solutions x for (MCT) and all feasible solutions (u, τ, w) for (MWD):*

(i) $\int_0^T f_i(t, x(t))dt$ *is ρ_i-SQIX with respect to function η for all $i \in L$*
(ii) $\int_0^T w(t)^T g(t, x(t))dt$ *is σ-QIX with respect to function η,*
(iii) $\sum \tau_i \rho_i + \sigma \geq 0.$
Then, the following can not be hold:

$$\int_0^T f_i(t, x(t))dt < \int_0^T f_i(t, u(t))dt \quad \text{for some} \quad i \in L \tag{21}$$

$$\int_b^T f_j(t, x(t))dt \leq \int_0^T f_j(t, u(t))dt, \quad \text{for all} \quad j \in L \tag{22}$$

Proof Suppose contrary to the result of theorem that (21 − 22) hold. Then (i) yields

$$\int_0^T f_i^0(t, u(t)); \eta(x(t), u(t))dt < -\rho_i \int_0^T d^2(t, x(t), u(t))dt, \tag{23}$$

for all $i \in L$. Multiplying each inequality of (23) by $\tau_i \geq 0$, and summing up for all $i \in L$, we obtain

$$\int_0^T \sum_{i=1}^r f_i^0(t, u(t)); \eta(x(t), u(t))dt < -\left(\sum_{i=1}^r \rho_i \tau_i\right) \int_0^T d^2(t, x(t), u(t))dt, \tag{24}$$

for all $i \in L$. Since x is feasible for (MCT) and (u, τ, w) is feasible for (MWD), it follows that

$$\int_0^T w(t)^T g(t, x(t))dt \leq \int_0^T w(t)^T g(t, u(t))dt. \tag{25}$$

Now by using (ii), we get

$$\int_0^T w(t)^T g^0(t, u(t); \eta(x(t), u(t))dt \leq -\sigma \int_0^T d^2(t, x(t), u(t))dt \tag{26}$$

adding (24) and (26),

$$\int_0^T \left\{\sum \tau_i f_i^0(t, u(t)); \eta(x(t), u(t)) + w(t)^T g^0(t, u(t), \eta(x(t), u(t)))\right\} dt <$$

$$-\left(\sum_{i=1}^{r}\tau_i\rho_i+\sigma\right)\int_0^T d^2(t,x(t),u(t))dt, \forall i\in L.$$

It follows from hypothesis (iii) that

$$\int_0^T\left(\sum_{i=1}^{r}\tau_i f_i^0(t,u(t));\eta(x(t),u(t))+w(t)^T g^0(t,u(t),\eta(x(t),u(t)))\right)dt<0. \quad (27)$$

This contradicts (31). Hence we have the result. □

Remark 4.6 The weak duality theorem also holds under the following different types of assumptions;

(a) $\int_0^T f_i(t,x(t))dt$ is ρ_i-QIX with respect to functions η for all $i\in L$ and $\int_0^T w(t)^T g^0(t,x(t))dt$ is σ-SQIX with respect to the same η and $\sum\tau_i\rho_i+\sigma\geq 0$.

(b) $\int_0^T f_i(t,x(t))dt$ is ρ_i-QIX with respect to functions η for all $i\in L$ and $\int_0^T w(t)^T g^0(t,x(t)dt$ is σ-QIX with respect to the same η , and $\sum\tau_i\rho_i+\sigma>0$.

Corollary 4.7 *Assume that weak duality theorem 4.5 holds between (MCT) and (MWD). If u^0 is a feasible solution for (MCT), and (u^0,τ^0,ω^0) is a feasible solution for (MWD). Then, u^0 is an efficient solution for (MCT) and (u^0,τ^0,ω^0) is an efficient solution for (MWD).*

Proof Suppose u^0 is not efficient for (MCT). Then there exists a feasible solution x for (MCT) such that

$$\int_0^T f_i(t,x(t))dt<\int_0^T f_i(t,u^0(t))dt \quad \text{for some} \quad i\in L,$$
$$\int_0^T f_i(t,x(t))dt\leq\int_0^T f_i(t,u^0(t))dt \quad \text{for all} \quad i\in L$$

This contradicts weak duality. Hence u^0 is feasible for MCT . Now suppose that(u^0,τ^0,ω^0) is not efficient for (MWD). Then there exists (u,τ,ω) feasible for (MWD) such that

$$\int_0^T\{f_i(t,u(t))dt>\int_0^T\{f_i(t,u^0(t))dt, \quad \text{for some} \quad i\in L$$
$$\int_0^T\{f_i(t,u(t))dt\geq\int_0^T\{f_i(t,u^0(t))dt, \quad \text{for all} \quad i\in L$$

This contradicts weak duality. Hence (u^0,τ^0,ω^0) is not efficient for (W2MCP). □

Theorem 4.8 *Let u^0 be efficient for (MCT) and assume that u^0 satisfies the constraint qualification (CQ) for $P_k(u^0)$ for at least one $k\in L$. Then there exist $\tau^0\in\mathbb{R}^r$ and piecewise smooth function $w^0: I\to\mathbb{R}^m$ such that (u^0,τ^0,w^0) is feasible for (MWD). If weak duality also holds between (MCT) and (MWD) then (u^0,τ^0,w^0) is efficient for (MWD).*

Proof Since u^0 satisfies the constraint qualification for at least one k, it follows from Lemma 3.2 that there exist $\tau^0 \in \mathbb{R}^r$ and piecewise smooth function $w^0 : I \to \mathbb{R}^m$ such that

$$0 \in \int_0^T \left\{ \sum_{i=1}^r \tau_i^0 (\partial_c f_i(t, u^0(t)) + w^0(t)^T \partial_c g(t, u^0(t))) \right\} dt \quad t \in [0, T] \tag{28}$$

$$w^0(t)^T g(t, u^0(t)) = 0, \quad a.e. \quad t \in [0, T] \tag{29}$$

$$w^0(t) \geq 0, \quad \tau_i^0 \geq 0, i = 1, \ldots, r \quad \sum_{i=1}^r \tau_i^0 = 1. \tag{30}$$

Now it follows from $(28-29)$, that (u^0, τ^0, w^0) is feasible for (MWD). Thus the result follows from Corollary 4.7. □

We end our paper with the following example which illustrates the Mond-Weir type dual.

Example 4.9 Consider the following multiobjective continuous-time problem:

$$(MCT) \min \left[\int_0^1 f_1(t, x(t))dt, \int_0^1 f_2(t, x(t))dt \right]$$
$$s.t. \quad g_i(t, x(t)) \leq 0, i \in M = \{1, 2, 3\}, a.e. \quad t \in [0, 1]$$

where

$$f_1(t, x(t)) := |x_1(t) - t|, \quad f_2(t, x(t)) := |x_2(t) - t|,$$
$$g_1(t, x(t)) := -x_1(t), g_2(t, x(t)) := -x_2(t), \quad g_3(t, x(t)) = x_1(t) + x_2(t) - 2t,$$

and $x : [0, 1] \to \mathbb{R}^2$ is defined by $x(t) = (x_1 t, x_2 t), x_1, x_2 \in \mathbb{R}$.

It can be easily verified that $x^*(t) = (t, t)$ is an efficient point of the problem (MCT) and $P_K(x^*)$ for $k = 1, 2$ satisfy constraint qualification. The necessary optimality condition is satisfied for $\tau_1 = \frac{1}{2}, \tau_2 = \frac{1}{2}, w_1^*(t) = 0, w_2^*(t) = 0, w_3^*(t) = t$, and

$$0 \in \int_0^1 \left\{ \sum_{i=1}^2 \tau_i \partial_c f_i(t, x^*(t)) + w(t)^T \partial_c g(t, x^*(t)) \right\} dt, \quad t \in [0, 1] \tag{31}$$

$$\int_0^1 w(t)^T g(t, x^*(t))dt \geq 0, \tag{32}$$

$$\sum_{i=1}^2 \tau_i = 1, \quad \tau_i \geq 0, \quad w_j(t) \geq 0, \quad j \in M, \quad a.e. \quad t \in [0, 1].$$

Hence (x^*, τ^*, w^*) is efficient for (MWD), where $x^*(t) = (t, t), \tau^* = (\frac{1}{2}, \frac{1}{2})$ and $\omega^*(t) = (0, 0, t)$.

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